The Weibel instability for bi-Maxwellian PDFs - relativistic existence conditions

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Outline

1. Kinetic theory of plasma instabilities
2. The Weibel instability
3. First results
4. Future work

Kinetic theory of plasma instabilities

- Particles and fields of a plasma can be described by the coupled Vlasov-Maxwell system of equations

  $\Rightarrow$ In general: Difficult!
- Describe plasma instablities by the linearized Vlasov equation for particle distribution function (PDFs) in one-particle phase space $f_a(\vec{x}, \vec{p}, t)$ for species a
- Fourier-Laplace transform to frequency space (k real $\Rightarrow \omega$ generally complex) to yield late term behavior of small amplitude fluctuations:

$$\delta f_a(\vec{x}, \vec{p}, t) = \int_{-\infty}^{\infty} \mathrm{d}^3 k \int_{0}^{\infty} \mathrm{d}\,\omega\, f_a(\vec{x}, \vec{p}, t) \exp\left[\imath \left(\vec{k} \cdot \vec{x} - \omega t\right)\right]$$

- Possible modes are given by solutions of the dispersion relation: $\Lambda(\vec{k}, \omega) = \det \Lambda_{ij} = 0$
- Maxwell operator $\Lambda_{ij} = \frac{c^2 k^2}{\omega^2} \left(\frac{k_i k_j}{k^2} - \delta_{ij} \right) + \psi_{ij}$
- Dielectric tensor $\psi_{ij} = \delta_{ij} + \frac{4\pi\imath}{\omega} \sigma_{ij}$
- Conductivity tensor σ_{ij} generally very complex function of the PDF and the phase speed z

The Weibel instability

- Transverse, weakly propagating plasma instability in collisionless, (temperature) anisotropic plasmas
- Dispersion relation known and evaluated in nonrelativistic limit in the limit of zero magnetic field by Schlickeiser et al. (2010)
- Now: covariant treatment to derive existence conditions for relativistic plasma particles
- Important for
  - Shockwaves in space
  - Gamma ray bursts
  - Particle beams through plasmas

The particle distribution function

Use temperature anisotropic bi-Maxwellian PDF

$$f(E,y) = \frac{1}{\pi^{3/2} A u_\parallel^3} \exp\left[-\frac{A\,(m_a c)^2}{u_\parallel^2}\left(E^2-1\right) + \frac{(A-1)\,(m_a c)^2}{u_\parallel^2} y^2\right]$$

- Variables introduced by Lerche, 1967: $y = \frac{p_\parallel}{m_a c}$, $E = \sqrt{1 + \frac{p_\parallel^2 + p_\perp^2}{m_a^2 c^2}}$
- Temperature anisotropy A, defined as $A := \frac{T_\parallel}{T_\perp} = \frac{u_\parallel^2}{u_\perp^2}$

Dispersion relation

- Dispersion relation from determinant of Maxwell operator $\det \Lambda_{ij} = 0$
- Form of the Maxwell operator from the dielectric tensor $\psi_{ij} = \delta_{ij} + \chi_{ij}$
- For transverse fluctuations it is given by

$$D = z^2\Lambda = z^2 - 1 - \frac{\pi A}{k^2 c^2} \sum_a \omega_{p,a}^2 \left(m_a c\right)^3 \times \int_1^\infty \mathrm{d}\,E \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} \mathrm{d}\,y \, \frac{E^2 - 1 - y^2}{y - Ez} \left(\frac{\partial f}{\partial y} - z \frac{\partial f}{\partial E} \right) = 0.$$

- Investigate this dispersion relation for bi-Maxwellian PDF given above

The following equation has to be investigated

Dispersion relation

Introduce the phase speed $z = \frac{\omega}{kc} = \frac{\Re e\omega}{kc} + \imath\frac{\Im m\omega}{kc} = R + \imath S$

$$D = z^2 - 1 - \frac{\pi A}{k^2 c^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \times$$

$$\int_1^\infty \mathrm{d}\,E \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} \mathrm{d}\,y \frac{E^2 - 1 - y^2}{y - Ez} \left\{ \frac{2\,(m_a c)^2}{\pi^{3/2} A u_\parallel^5} \left[(A-1)\,y + AzE\right] \times \right.$$

$$\left. \exp\left[-\frac{A\,(m_a c)^2}{u_\parallel^2}\left(E^2 - 1\right) + \frac{(A-1)\,(m_a c)^2}{u_\parallel^2} y^2 \right] \right\}$$

First results

After straightforeward but messy calculus we are left with

Real and Imaginary part of the dispersion relation

Reminder: Phase speed $z = R + \imath S$, set $R = 0$..

$$\mathfrak{Re} D(R=0,S) = -S^2 - 1 + \frac{1}{\pi^{1/2} u_{\|}^3 k^2 c^2} \sum_a \omega_{p,a} (m_a c)^3 \times$$

$$\int_1^\infty \mathrm{d}\,E\,\left(E^2-1\right)\left\{\exp\left[\frac{-(m_a c)^2}{u_{\|}^2}\left(E^2-1\right)\right] + \exp\left[\frac{(1-2A)(m_a c)^2}{u_{\|}^2}\left(E^2-1\right)\right]\right\}$$

and

$$\mathfrak{Im} D(R=0,S) = -\frac{2}{\pi^{1/2} u_{\|}^5 k^2 c^2} \sum_a \omega_{p,a} (m_a c)^5 \times$$

$$\int_1^\infty \mathrm{d}\,E\,\left\{\left(AES - E^2 S^2\right)\frac{\sqrt{\pi}\,u_{\|}}{\sqrt{A-1}\,m_a c}\exp\left[\frac{-A(m_a c)^2}{u_{\|}^2}\left(E^2-1\right)\right]\mathrm{erfi}\left[\frac{(A-1)^{1/2}\,m_a c}{u_{\|}^2}\left(E^2-1\right)\right]\right.$$

$$\left. + \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} \mathrm{d}\,y\left[\frac{1}{y-ES}+\frac{1}{y+ES}\right]\exp\left[\frac{(A-1)(m_a c)^2}{u_{\|}^2}y^2 + \frac{A(m_a c)^2}{u_{\|}^2}\left(E^2-1\right)\right]\right\}$$

Result for $\Re e D$

First existence condition here:

Integral in $\Re e D(R = 0, S)$ converges only for $A > 0.5$.

The integral in $\Re e D(R = 0, S)$ can be done analytically:

$$\Re e D(R=0,S) = -S^2 - 1 + \frac{1}{2\,u_{\|}^2\,k^2\,c^2} \sum_a \omega_{p,a}\,(m_a\,c)^2 \times$$

$$\left\{ \frac{u_{\|}^2}{2\,(m_a\,c)^2} \exp\left[\frac{(m_a\,c)^2}{u_{\|}^2}\right] \left[1 - \mathrm{erf}\left[\frac{(m_a\,c)}{u_{\|}}\right]\right] + \frac{u_{\|}}{(m_a\,c)} - \exp\left[\frac{(m_a\,c)^2}{u_{\|}^2}\right] \left[1 - \mathrm{erf}\left[\frac{(m_a\,c)}{u_{\|}}\right]\right] \right.$$

$$+ \frac{u_{\|}^2}{2\,(m_a\,c)^2\,(1-2A)} \exp\left[\frac{-(1-2A)\,(m_a\,c)^2}{u_{\|}^2}\right] \left[1 - \mathrm{erf}\left[\frac{(1-2A)^{1/2}\,(m_a\,c)}{u_{\|}}\right]\right] + \frac{u_{\|}}{(m_a\,c)\,(1-2A)}$$

$$\left. - \frac{1}{1-2A} \exp\left[\frac{-(1-2A)\,(m_a\,c)^2}{u_{\|}^2}\right] \left[1 - \mathrm{erf}\left[\frac{(1-2A)\,(m_a\,c)}{u_{\|}}\right]\right] \right\} = 0.$$

Work in progress and future ideas

- What about the imaginary part? $\Rightarrow$ Numerical analysis.
- Use relativistically correct generalisation of bi-Maxwell PDF
- Do the same for bi-kappa PDF
- Similar investigations for other instabilities, e.g. mirror, firehose modes